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Recounting Dyons in $\mathcal{N} = 4$ String Theory

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Abstract

A recently discovered relation between 4D and 5D black holes is used to derive weighted BPS black hole degeneracies for 4D $\mathcal{N} = 4$ string theory from the well-known 5D degeneracies. They are found to be given by the Fourier coefficients of the unique weight 10 automorphic form of the modular group $Sp(2,\mathbb{Z})$. This result agrees exactly with a conjecture made some years ago by Dijkgraaf, Verlinde and Verlinde.

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A general D0-D2-D4-D6 black hole in a 4D IIA string compactification has an M-theory lift to a 5D black hole configuration in a multi-Taub-NUT geometry. This observation was used in [1] to derive a simple relation between 5D and 4D BPS black hole degeneracies. For the case of $K3 \times T^2$ compactification, corresponding to $\mathcal{N} = 4$ string theory, the relevant 5D black holes were found in [2,3] and the degeneracies are well known. In this paper we translate this into an exact expression for the 4D degeneracies, which turn out to be Fourier expansion coefficents of a well-studied weight 10 automorphic form Φ of the modular group of a genus 2 Riemman surface [4,5].

Almost a decade ago an inspired conjecture was made [5] by Dijkgraaf, Verlinde and Verlinde for the 4D degeneracies of $\mathcal{N} = 4$ black holes, and this was shown to pass several consistency checks. We will see that our analysis precisely confirms their old conjecture.

$\mathcal{N} = 4$ string theory in four dimensions can be obtained from IIA compactification on $K3 \times T^2$. The duality group is conjectured to be

$$SL(2;\mathbb{Z}) \times SO(6, 22;\mathbb{Z}). \tag{1}$$

The first factor may be described as an electromagnetic S-duality which acts on electric charges $q_{e\Lambda}$ and magnetic charges q_m^Λ, $\Lambda = 0, ..27$ transforming in the 28 of the second factor. For the electric objects, we may take

$$q_e = (q_0; q_A; q_{23}; q_j), \tag{2}$$

where q_0 is D0-charge, $q_A, A = 1, ...22$ is $K3$-wrapped D2 charge, q_{23} is $K3$-wrapped D4 charge, and $q_i, i = 24, ...27$ are momentum and winding modes of $K3 \times S^1$-wrapped NS5 branes. The magnetic objects are 24 types of D-branes which wrap $T^2 \times$ (K3 cycle) and 4 types of F-string T^2 momentum/winding modes.

Now consider a black hole corresponding to a bound state of a single D6 brane with D0 charge q_0, $K3$-wrapped D2 charge q_A, and T^2-wrapped D2 charge q^{23}:

$$q_m = (1; q^A = 0; q^{23}; q^i = 0), \quad q_e = (q_0; q_A; q_{23} = 0; q_i = 0) \tag{3}$$

The duality invariant charge combinations are

$$\frac{1}{2}q_e^2 = \frac{1}{2}C^{AB} q_A q_B, \quad \frac{1}{2}q_m^2 = q^{23}, \quad q_e \cdot q_m = q_0 \tag{4}$$

where C^{AB} is the intersection matrix on $H^2(K3;\mathbb{Z})$.

By lifting this to M-theory on Taub-NUT, it was argued in [1] that the BPS states of this system are the same as those of a 5D black hole in a $K3 \times T^2$ compactification, with T^2-wrapped M2 charge $\frac{1}{2}q_m^2$, $K3$-wrapped M2 charge q_A and angular momentum $J_L = q_0/2$. We now use one of the compactification circles to interpret the configuration as IIA on $K3 \times S^1$ with $\frac{1}{2}q_m^2$ F-strings winding S^1 and q_A D2-branes. T-dualizing the S^1 yields q_A D3-branes carrying momentum $\frac{1}{2}q_m^2$. This is then U-dual to a Q_1 D1 branes and Q_5 D5 branes on $K3 \times S^1$ with

$$N \equiv Q_1 Q_5 = \frac{1}{2}q_e^2 + 1 \tag{5}$$

angular momentum[1]

$$J_L = \frac{1}{2} q_e \cdot q_m \tag{6}$$

and left-moving momentum along the S^1:

$$L_0 = \frac{1}{2}q_m^2. \tag{7}$$

Hence, with the above relations between parameters, according to [1] the 4D degeneracy of states with charges (3) and 5D degeneracies are related by

$$d_4(1; 0; q^{23}; 0|q_0; q_A; 0; 0) = (-1)^{q_0} d_5\left(q^{23}, q_A; \frac{q_0}{2}\right). \tag{8}$$

The extra factor of $(-1)^{q_0}$ comes from the extra insertion of $(-1)^{2J_L}$ in the definition of the 5D index. Since the degeneracies are U-dual we may also write[2]

$$d_4(q_m^2, q_e^2, q_e \cdot q_m) = (-1)^{2J_L} d_5(L_0, N, J_L) = (-1)^{q_e \cdot q_m} d_5\left(\frac{1}{2}q_m^2, \frac{1}{2}q_e^2 + 1, \frac{1}{2}q_e \cdot q_m\right). \tag{9}$$

Here and elsewhere in this paper by "degeneracies," in a slight abuse of language, we mean the number of bosons minus the number of fermions of a given charge, and the center-of-mass multiplet is factored out.

[1] One should keep in mind that J_L is half the R-charge F_L [3], and is hence takes values in $\frac{1}{2}\mathbb{Z}$.

[2] Note that d_n denotes fixed-charge degeneracies and does not involve a sum over U-duality orbits.

Of course these microscopic BPS degeneracies d_5 of the D1-D5 system are well known [2,3]. Their main contribution comes from the coefficients in the Fourier expansion of the elliptic genus of $\mathrm{Hilb}^N(K3)$:

$$\chi_N(\rho,\nu) = \sum_{L_0,J_L} d_5'(L_0,N,J_L) e^{2\pi i(L_0\rho+2J_L\nu)} \tag{10}$$

It is shown in [6] that the weighted sum of the elliptic genera has a product representation:

$$\sum_{N\geq 0} \chi_N(\rho,\nu) e^{2\pi i N\sigma} = \frac{1}{\Phi'(\rho,\sigma,\nu)} \tag{11}$$

where Φ' is given by

$$\Phi'(\rho,\sigma,\nu) = \prod_{k\geq 0, l>0, m\in\mathbb{Z}} (1-e^{2\pi i(k\rho+l\sigma+m\nu)})^{c(4kl-m^2)}, \tag{12}$$

with $c(4k-m^2) = d_5'(k,1,m)$ the elliptic genus coefficients for a single $K3$ as given in [7].[3]

Equation (11) is the generating function for BPS states of CFTs on $\mathrm{Hilb}^N(K3)$ in the D5 worldvolume. However it does not quite give the degeneracies needed in (9) because it leaves out the decoupled contribution from the elliptic genus of a single fivebrane. This remains even when $N=0$ and there are no D1 branes at all. (By U-duality, we are free to view the system as a single fivebrane and N D1 branes.) Using the U-dual relation of a $K3$ -wrapped D5 brane to a fundamental heterotic string, the elliptic genus, not including the center of mass contribution, is [8,9]

$$Z_0(\nu,\rho) = (e^{\pi i\nu}-e^{-\pi i\nu})^{-2} e^{-2\pi i\rho} \prod_{n\geq 1} (1-e^{2\pi i(n\rho+\nu)})^{-2}(1-e^{2\pi i(n\rho-\nu)})^{-2}(1-e^{2\pi i n\rho})^{-20}. \tag{13}$$

This shifts Φ' to

$$\frac{1}{\Phi'(\rho,\sigma,\nu)} \to \frac{Z_0(\nu,\rho)}{\Phi'(\rho,\sigma,\nu)} = \frac{e^{2\pi i\sigma}}{\Phi(\rho,\sigma,\nu)} \tag{14}$$

where $\Phi(\rho,\sigma,\nu)$ has a product representation

$$\Phi(\rho,\sigma,\nu) = e^{2\pi i(\rho+\sigma+\nu)} \prod_{(k,l,m)>0} \left(1-e^{2\pi i(k\rho+l\sigma+m\nu)}\right)^{c(4kl-m^2)} \tag{15}$$

where $(k,l,m)>0$ means that $k,l\geq 0$, $m\in\mathbb{Z}$ and in the case $k=l=0$, the product is only over $m<0$. $\Phi(\rho,\sigma,\nu)$ is the unique automorphic form of weight 10 of the modular

[3] Note $c(-1)=2$, $c(0)=20$, and $c(n)=0$ for $n\leq -2$.

group $Sp(2,\mathbb{Z})$ and was studied in [4]. The 5D BPS degeneracies are then the Fourier coefficients in

$$\sum_{L_0,N,J_L} d_5(L_0,N,J_L)e^{2\pi i(L_0\rho+(N-1)\sigma+2J_L\nu)} = \frac{1}{\Phi(\rho,\sigma,\nu)}. \tag{16}$$

Inserting the 4D-5D relation (9), (16) agrees with the formula proposed in [5] for the microscopic degeneracy of BPS black holes of $\mathcal{N}=4$ string theory – up to an overall factor of $(-1)^{q_e\cdot q_m}$.[4]

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[4] Note that the formula of [5] was manifestly invariant under the duality group (1), and the extra factor of $(-1)^{q_e\cdot q_m}$ does not spoil this. Invariance of $(-1)^{q_e\cdot q_m}$ under $SO(6,22;\mathbb{Z})$ follows by construction, and invariance under $SL(2;\mathbb{Z})$ follows from the S-duality transformations of the charges.

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